Form Factors in Quantum Electrodynamics*

S. D. DRELL AND F. ZACHARIASEN
Stanford University, Stanford, California

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The electromagnetic form factors of an electron in pure quantum electrodynamics are analyzed with the techniques of dispersion relations. The viewpoint is adopted here that no subtractions are required in the construction of dispersion relations for the electromagnetic vertex. This leads to coupled integral equations for the form factors in terms of other physical amplitudes; electron-positron scattering, for example. The relation between this and the usual perturbation approach to quantum electrodynamics, and the validity and consequences of the "no-subtraction" philosophy, are discussed.

I. INTRODUCTION

IN constructing dispersion relations for any process it is an open question whether or not subtraction constants are required. In practice, in the analysis of pion-nucleon scattering and photoproduction by means of dispersion relations,[1] the assumption that no such constants exist has led to reasonably good agreement with experiment. As a result it is tempting to suggest that it is a general rule that such arbitrary constants are never present.[2] If such a "no-subtraction" philosophy is combined with the assumption that dispersion relations hold for all amplitudes, then one obtains an infinite set of coupled homogeneous integral equations connecting these amplitudes.

We should like to explore here the no-subtraction philosophy, its validity and consequences. For this purpose, we select quantum electrodynamics (QED), and in particular, we study the behavior of the electromagnetic form factors of an electron in pure QED. There are several reasons for this choice. The first is that pure QED refers only to one coupling, and one coupling constant. The second is that experimentally valid solutions, at least in the low-energy region, are known to be obtainable through perturbation theory. Finally, we fix on the form factors because they describe a process with only three external particles and hence satisfy a simple type of dispersion relation. Our purpose then is to study the agreement with experiment of the form factors obtained from the no-subtraction philosophy. That is, we ask if it is possible to construct form factors which vanish for infinite momentum transfers, have the analyticity properties required for the existence of dispersion relations, and also agree with the usual perturbation theory, and hence experiment, at low-momentum transfers.

It is convenient to begin with a brief discussion of the usual perturbation approach to QED, and then to compare it with the dispersion theoretic approach.

The conventional procedure of calculation in QED may be summarized as follows, with particular reference to the form factors. The complete vertex contains an infinite number of Feynman graphs (Fig. 1). Corresponding to this expansion, one may write the vertex as

$$F_1(q^2)\gamma_\mu+F_2(q^2)\sigma_{\mu\nu}q_\nu, \tag{1}$$

with

$$F_1(q^2)=e_0+a_1(q^2)e_0^3+\cdots,$$

and

$$F_2(q^2)=b_1(q^2)e_0^3+\cdots, \tag{2}$$

where e_0 is the "bare" charge and q_μ is the four-momentum transfer at the vertex.

The "observable" charge is now *defined* by

$$e=e_0+a_1(0)e_0^3+\cdots, \tag{3}$$

and is experimentally found to be small, $e^2/4\pi=1/137$. "Solving" Eq. (3) for e_0 in terms of e and inserting in Eqs. (2), one may write

$$\begin{aligned}F_1(q^2)&=e+c_1(q^2)e^3+\cdots,\\ F_2(q^2)&=d_1(q^2)e^3+\cdots,\end{aligned} \tag{4}$$

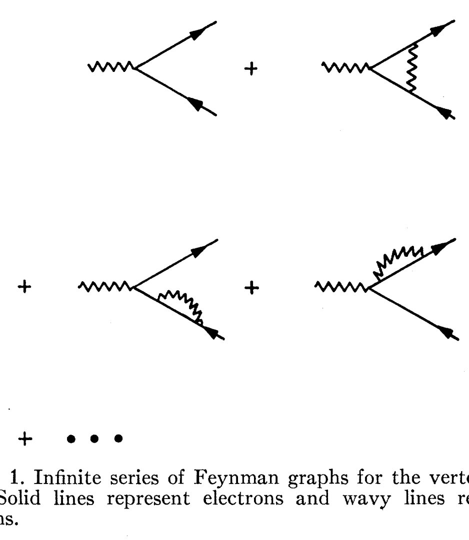

FIG. 1. Infinite series of Feynman graphs for the vertex function. Solid lines represent electrons and wavy lines represent photons.

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[1] Chew, Goldberger, Low, and Nambu, Phys. Rev. **106**, 1337, 1345 (1957).

[2] Chew, Gasiorowicz, Karplus, and Zachariasen (to be published). This point of view is mentioned in this reference though a subtraction was actually made in the calculation of the nucleon's charge form factor. Recently G. F. Chew has discussed the no-subtraction view of the nucleon structure factors. We wish to thank M. Gell-Mann for a stimulating discussion of the implications of the no-subtraction philosophy.

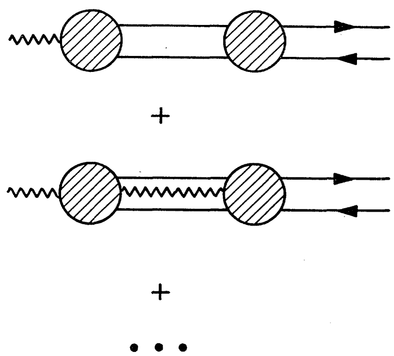

FIG. 2. Infinite series of "dispersion graphs" for the vertex function.

with $c_i(0)=0$, and with all expansion coefficients in Eqs. (4) finite. In practice, of course, infinite expansion coefficients appear in Eqs. (2). Because of the smallness of $e^2/4\pi$, Eqs. (4) have been found to form a useful expansion for low-momentum transfers.[3] They no longer have a term by term correspondence with the Feynman graphs of Fig. 1. In particular e corresponds to the first graph of Fig. 1, plus the sum of the values of all succeeding graphs at $q^2=0$.

The dispersion theory procedure of calculation may be described as expressing the form factors as an infinite series of "dispersion graphs." This is seen in Fig. 2. To this series of graphs there corresponds a series of functions calculated by well-defined rules,

$$[\mathrm{Im}F_1(q^2)]_{e\bar{e}}+[\mathrm{Im}F_1(q^2)]_{e\bar{e}\gamma}+\cdots,$$

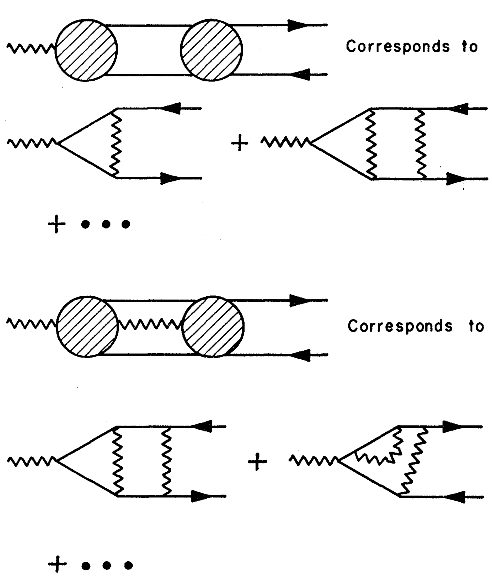

FIG. 3. Correspondence of a set of Feynman graphs to a dispersion graph.

[3] It is indicated by radiative correction calculations that the relevant expansion parameter is $(e^2/4\pi)\ln(q^2/m^2)$.

and similarly for F_2. (The correspondence of the function to its diagram is indicated by writing the intermediate state as a subscript.) The form factor is then obtained from the dispersion integral

$$\begin{aligned}F_1(q^2)&=\frac{1}{\pi}\int_0^\infty\frac{dq'^2}{q'^2-q^2-i\epsilon}\sum_n[\mathrm{Im}F_1(q'^2)]_n\\&\equiv\sum\nolimits_n[F_1(q^2)]_n.\end{aligned}\tag{A}$$

According to our assumption the integral in Eq. (A) exists,[4] so that no subtraction is required. If this assumption is not made, the dispersion integral may be given in the form with the charge e appearing explicitly as a subtraction constant:

$$\begin{aligned}F_1(q^2)&=e+\frac{q^2}{\pi}\int_0^\infty\frac{dq'^2}{q'^2(q'^2-q^2-i\epsilon)}\sum_n[\mathrm{Im}F_1(q'^2)]_n\\&=e+\sum\nolimits_n\{[F_1(q^2)]_n-[F_1(0)]_n\}.\end{aligned}\tag{B}$$

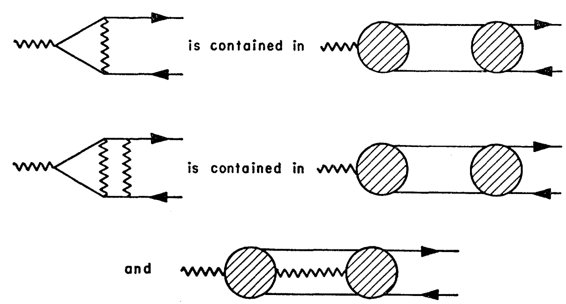

FIG. 4. Relation of a Feynman graph to dispersion graphs.

It is of immediate interest to inquire into the connection between the dispersion description of the form factor and the perturbation description. To make this relation clear, it is convenient to take the diagrams in both cases literally and to establish a correspondence between them. To each dispersion graph there corresponds a large and well-defined set of Feynman graphs, as indicated in Fig. 3.

Similarly all Feynman graphs, except for one, fall into the general types of dispersion graphs, as indicated in Fig. 4.[5] The one exception is the lowest order Feynman graph, Fig. 5, which we discuss now. It may be said to be included in the subtraction constant e if dispersion relations of type B are used. However, it is our intention in this paper to study the consequences of the assumption that dispersion relations of type (A) are applicable, in which case no term in $\sum_n[F_1(q^2)]_n$ explicitly contains Fig. 5. Now dispersion relations of type (A) imply that e is expressed in terms of itself, i.e.,

[4] Lehmann, Symanzik, and Zimmerman, Nuovo cimento **2**, 425 (1955).

[5] It is evident in Figs. 3 and 4 that the same Feynman graph contributes to more than one dispersion graph and vice versa.

$e=f(e)$. This follows directly from (A),

$$e=\sum_n [F_1(0)]_n, \tag{5}$$

when we recall that each $(F_1)_n$, by means of a dispersion relation, is itself expressible as a combination of amplitudes of physical processes which connect with each other, and which connect in particular, back to $F_1(q^2)$. For example, the first graph in Fig. 2, shows that $(F_1)_{e\bar{e}}$ is related to F_1 times the $e\bar{e}$ scattering amplitude; this scattering amplitude is related to $e^2/4\pi$ again, via the graph in Fig. 6, plus other terms. According to Eq. (5) then, the lowest order Feynman graph is given by a linear combination of many other Feynman graphs which involve e to higher powers. Since each of these graphs is included in the dispersion relations, we may consider that the lowest-order perturbation graph is included as well.

Equation (5) may be looked on in to ways. On one hand, one may say that Eq. (5) uniquely determines e. According to this view, the requirement that the form factors vanish at infinite momentum transfer forces a new condition on the field theory, not present in the usual description, which can be satisfied only for a particular value of e. A second point of view is that Eq. (5) is an identity in e. Then, as in the usual formulation of QED, a solution exists for any value of e. Another way of saying this is that the dispersion relations with no subtractions form an infinite set of coupled homogeneous integral equations for all conceivable physical amplitudes, in particular the charge form factor. The two points of view described above may then be expressed as follows: either this set of equations has a unique solution, or it does not. In the first case, e is uniquely determined; in the second case, the requirement that $F_1(0)=e$ picks out a class (perhaps one) of solutions; once this condition has been imposed, the equation becomes an identity in e.

At present, it is not known definitely which alternative applies. For our purposes, this is irrelevant, since, as will become clear later, our approximation is such that the imposition of condition (5) can be used only to obtain information about the form factor, and not about the charge itself.

Since it is in practice not possible to calculate $[\mathrm{Im}F_1(q^2)]_n$ for all intermediate states, we must work with an approximation to Eq. (A). We do not expand the form factors in a power series in e^2. It appears in out final result, in fact, that such an expansion is indeed impossible.[6] The nature of our approximation will be an

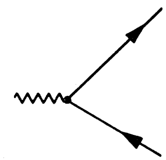

FIG. 5. Lowest order Feynman graph for the electromagnetic vertex.

expansion in intermediate states which contribute to $F_1(q^2)$, i.e., an expansion in terms of dispersion graphs instead of Feynman graphs. We note here a similarity to the analogous expansion used by Chew and Low in meson theory in calculating the physical amplitudes of the meson-nucleon interaction. In particular we retain only the state $n=e\bar{e}$. The infinite set of coupled integral equations is then reduced to equations coupling the form factors to the electron-positron scattering amplitude and to each other. In the dispersion relation for the $e\bar{e}$ scattering amplitude, we shall keep only the contribution from the pole, which is equvalent to renormalized Born approximation, except at threshold and zero scattering angle where an exact treatment is required to avoid the infrared problem. This description of $e\bar{e}$ scattering agrees well with experiments in the low-energy region.

II. DISPERSION RELATIONS

The dispersion relations upon which we base our discussions are the following:

$$F_{1,2}(q^2)=\frac{1}{\pi}\int_0^\infty \frac{dq'^2}{q'^2-q^2-i\epsilon}\,\mathrm{Im}F_{1,2}(q'^2), \tag{6}$$

where F_1 and F_2 are defined as the usual electromagnetic form factors:

$$\langle p_+p_-^{(-)}|j_\mu(0)|0\rangle=(4E_{p+}E_{p-})^{-\frac{1}{2}} \times(\bar{u}_{p-}|F_1(q^2)\gamma_\mu+F_2(q^2)\sigma_{\mu\nu}q_\nu|v_{p+}). \tag{7}$$

We use the following notation: $p_+=(E_{p+},\mathbf{p}_+)$ and $p_-=(E_{p-},\mathbf{p}_-)$ represent the four-momenta of the positron and electron which are produced in state $|p_+,p_-^{(-)}\rangle$ with incoming boundary conditions by $j_\mu(0)$, the complete electromagnetic current operator evaluated at $x_\mu=0$, operating on the physical vacuum state $|0\rangle$. The corresponding Dirac spinors for the positron and electron are v_{p+} and u_{p-} which are taken to be normalized to $\bar{u}_pu_p=-\bar{v}_pv_p=2m_e$ and which satisfy the equations $(p_--m_e)u_{p-}=0$; $(p_++m_e)v_{p+}=0$. Here we choose

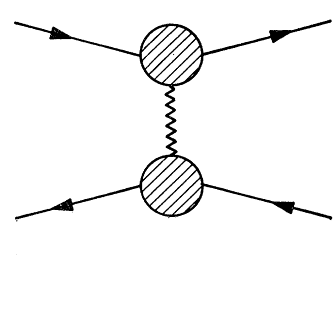

FIG. 6. Contribution to $e\bar{e}$ scattering amplitude proportional to e^2.

[6] In explicitly evaluating e from Eq. (5), one might first be tempted to expand the functions $[F_1(0)]_n$ in powers of e^2. This would lead to a series $e=p_1(0)e^3+p_2(0)e^5+\cdots$. It is clear that one should not expect such a series to be a reasonable approximation even though an expansion of the type indicated in Eq. (4) is in fact useful. This is analogous to an attempt to expand 1/137 in powers of 1/137 starting with $(1/137)^3$. It is however, a reasonable expansion to write 1/150, for example, in such a series starting with $(1/137)^1$.

$\gamma_\mu=(\beta,\beta\boldsymbol{\alpha})$ and $\sigma_{\mu\nu}=\gamma_\mu\gamma_\nu-\gamma_\nu\gamma_\mu$. Finally $q_\mu=(p_+)_\mu+(p_-)_\mu$ represents the four-momentum transfer and our metric is such that $q^2=q_0^2-\mathbf{q}^2$.

The properties of F_1 and F_2 which are required by Eq. (6) are that F_1 and F_2 are analytic everywhere in the q^2 plane with a branch cut along the real axis from $q^2=0$ to ∞, and that F_1 and F_2 vanish for large q^2. A reality condition

$$F_{1,2}(q^2)=F_{1,2}(q^{2*})^* \tag{8}$$

is also seen to hold for functions which satisfy relations of this type. This condition that F_1 and F_2 are real on the real q^2 axis (except of course on the branch line) also follows on physical grounds from the assumption that *QED* is invariant under time reversal.

We offer here no rigorous proof of dispersion relations Eq. (6).[2,7] However, the indicated branch cut and reality property for F_1 and F_2 may be seen to follow directly upon application of the contraction rule to the QED vertex, Eq. (7). Thus, combining the electron state vector with the operator in Eq. (7), one obtains the relation†

$$\langle p_+p_-^{(-)}|j_\mu(0)|0\rangle=-i\int d^4x(2E_{p_-})^{-\frac{1}{2}}e^{ip\cdot x}\times\eta(x_0)\langle p_+|[\bar{u}_{p_-}j_e(x),j_\mu(0)]|0\rangle, \tag{9}$$

where

$$(i\gamma_\mu\nabla_\mu-m_e)\psi_e(x)=j_e(x)$$

and

$$\eta(x_0)=1,\quad x_0>0$$
$$=0,\quad x_0<0.$$

Inserting a complete set of states and integrating one reduces this to a form which directly shows the branch cut:

$$\langle p_+p_-^{(-)}|j_\mu(0)|0\rangle=-(2E_{p_-})^{-\frac{1}{2}}\sum_n\left\{\frac{\langle p_+|\bar{u}_{p_-}j_e(0)|n^{(\pm)}\rangle\langle n^{(\pm)}|j_\mu(0)|0\rangle(2\pi)^3\delta^3(\mathbf{p}_++\mathbf{p}_--\mathbf{p}_n)}{E_n-E_{p+}-E_{p-}-i\epsilon}-\frac{\langle p_+|j_\mu(0)|n^{(\pm)}\rangle\langle n^{(\pm)}|\bar{u}_{p_-}j_e(0)|0\rangle(2\pi)^3\delta^3(\mathbf{p}_-+\mathbf{p}_n)}{E_n+E_{p_-}}\right\}. \tag{10}$$

The second term of Eq. (10) has no singularity as is readily seen in a coordinate system in which the electron is at rest. The first term is singular whenever

$$q=p_++p_-=p_n,$$

and the minimum value of q^2 satisfying this condition is 0, corresponding to the threshold for production of three photons.

The reality condition, Eq. (8), may be proved directly by comparing Eqs. (7) and (9) in the coordinate system in which the positron is at rest with the complex conjugate of the corresponding expression obtained from Eq. (7) by combining with the operator the positron state vector instead of the electron state vector and transforming to the system with the electron at rest. There results

$$\begin{aligned}F_1(q^2)\gamma_\mu&=F_1(q^{2*})^*\gamma_0\gamma_\mu{}^\dagger\gamma_0,\\F_2(q^2)\sigma_{\mu\nu}&=-F_2(q^{2*})^*\gamma_0\sigma_{\mu\nu}{}^\dagger\gamma_0,\end{aligned} \tag{11}$$

or $F_1(q^2)=F_1(q^{2*})^*$ and $F_2(q^2)=F_2(q^{2*})^*$.

Using reality condition Eq. (8), we have a direct relation betweem $\mathrm{Im}F_{1,2}$ and the discontinuity across the branch line; explicitly, by Eqs. (7) and (10),

$$\begin{aligned}(4E_{p+}E_{p-})^{-\frac{1}{2}}(\bar{u}_{p_-}|\mathrm{Im}F_1(q^2)\gamma_\mu+\mathrm{Im}F_2(q^2)\sigma_{\mu\nu}q_\nu|\bar{u}_{p+})\\=-\tfrac{1}{2}(2E_{p_-})^{-\frac{1}{2}}\textstyle\sum_n(2\pi)^4\delta^4(p_n-q)\\\times\langle p_+|\bar{u}_{p_-}j_e(0)|n^{(\pm)}\rangle\langle n^{(\pm)}|j_\mu(0)|0\rangle\\\equiv(4E_{p+}E_{p-})^{-\frac{1}{2}}A_\mu(p_+,p_-).\end{aligned} \tag{12}$$

The imaginary parts of F_1 and F_2 thus appear in Eq. (12) as a sum of contributions from various intermediate states. The contribution of each state n vanishes for q^2 less than the square of the total mass corresponding to that state.

Separating off the contributions from the electron-positron intermediate state (the dispersion graph in Fig. 3), we write

$$\mathrm{Im}F_{1,2}=\mathcal{G}_{e\bar{e}}{}^{1,2}+\sum_{n\neq e\bar{e}}\mathcal{G}_n{}^{1,2}. \tag{13}$$

Choosing one-half the sum of ingoing plus outgoing boundary conditions for the intermediate state sum, we preserve the correct reality condition for each individual contribution to $A_\mu(p_+,p_-)$, and hence of $A_\mu(p_+,p_-)$ itself in any approximation which neglects contributions of some states n. The contribution of the electron-positron intermediate state in Eq. (12) is proportional to the form factors themselves, the proportionality factors being essentially just the physical electron-positron scattering phase shifts. Thus, we write, with a and b representing partial scattering amplitudes,

$$\begin{aligned}\mathrm{Im}F_{1,2}&=\mathrm{Re}(a_{1,2}F_{1,2}{}^*)+\mathrm{Re}(b_{1,2}F_{2,1}{}^*)+\sum_{n\neq e\bar{e}}\mathcal{G}_n{}^{1,2}\\&=\frac{\mathrm{Re}a_{1,2}}{1-\mathrm{Im}a_{1,2}}\mathrm{Re}F_{1,2}+\frac{\mathrm{Re}(b_{1,2}F_{2,1}{}^*)}{1-\mathrm{Im}a_{1,2}}\\&\quad+\sum_{n\neq e\bar{e}}\mathcal{G}_n{}^{1,2}/(1-\mathrm{Im}a_{1,2}).\end{aligned} \tag{14}$$

Inserting this into Eq. (6), we obtain an integral equa-

[7] Bremermann, Oehme, and Taylor, Phys. Rev. **109**, 2178 (1958); Bogoliubov, Medvedov, and Polivanov, Institute for Advanced Study Notes, Princeton, 1956 (unpublished).

† We have dropped a one-time commutator in Eq. (9) because it is a constant in the dispersion variable and does not contribute to our discussion.

tion from which the functions $F_1(q^2)$ and $F_2(q^2)$ may be determined. In particular, the electric charge emerges as $F_1(0)$ and the anomalous magnetic moment of the electron as $-2F_2(0)$.

Our problem is then reduced to the nontrivial evaluation of a, b, and $\mathcal{G}_n$ in the right-hand side of Eq. (14).

III. EVALUATION OF ABSORPTIVE AMPLITUDE

In this section, we calculate the absorptive amplitude in the approximation of neglecting all but the electron-positron state, $n=e\bar{e}$. In support of this, we refer back to Sec. I and note that the amplitude of all higher states has a threshold dependence down by a factor of 1/137. Since, however, the dispersion integral ranges over all values of momentum transfer and experimental information on QED is available only up to ~ 200 Mev/c,[8] neglect of the higher states in Eqs. (13) and (14) is not a defensible approximation. It is, however, an interesting one to pursue in order to obtain, perhaps, a qualitative insight into the behavior of the form factors. After carrying out the calculations in this approximation, we shall discuss this point further.

We turn now to the calculation of amplitudes $a_{1,2}$. As pointed out above Eq. (14), these are related to the $e\bar{e}$ scattering amplitude, and in particular to scattering in the ${}^3S_1$ and ${}^3D_1$ states, the only states which a photon can produce, according to angular momentum, parity, and charge conjugation selection rules. With neglect of the annihilation channel for the $e\bar{e}$ interaction, one can follow the discussion of Blatt and Biedenharn[9] for two-channel reactions and relate these amplitudes to the two eigenphase shifts and one mixing parameter for the ${}^3S_1$ and ${}^3D_1$ states.

We do this as follows: First, from Eq. (12), the absorptive part is written as

$$[A_\mu(p_+,p_-)]_{e\bar{e}}=\tfrac{1}{2}[A_\mu^{+}(p_+,p_-)]_{e\bar{e}}+\tfrac{1}{2}[A_\mu^{-}(p_+,p_-)]_{e\bar{e}},$$

with

$$[A_\mu^{\pm}(p_+,p_-)]_{e\bar{e}}=-\left(\frac{E_{p+}}{2}\right)^{\frac{1}{2}}\int\frac{d^3q_+}{(2\pi)^3}\int\frac{d^3q_-}{(2\pi)^3}\times\sum_{\text{spins}}(2\pi)^4\delta^4(p_++p_--q_+-q_-)\times\langle p_+|\bar{u}_{p-}j_e(0)|q_+q_-^{(\pm)}\rangle\langle q_+q_-^{(\pm)}|j_\mu(0)|0\rangle. \quad (15)$$

We may then write for the electron-positron scattering amplitude[9]

$$(2E_{p-})^{-\frac{1}{2}}\langle p_+|\bar{u}_{p-}j_e(0)|q_+q_-^{(+)}\rangle=-\frac{8\pi^2}{pE_p}\sum_{Mll'\mu\mu'}(i)^{l-l'-1}C_{l'1}(1M\mu'm_s')\times C_{l1}(1M\mu s)(\delta_{l'l}-S_{l'l}^{J=1,S=1})\times Y_{l'\mu'}^*(\Omega_p)Y_{l\mu}(\Omega_q), \quad (16)$$

[8] S. D. Drell, Ann. Phys. (to be published).

[9] J. M. Blatt and L. C. Biedenharn, Jr., Revs. Modern Phys. **24**, 258 (1952).

in terms of the center-of-mass system, with $E_p=E_{p+}=E_{p-}$ and $\mathbf{p}=\mathbf{p}_+=-\mathbf{p}_-$; Ω_p and Ω_q are the angles of the final and initial relative momenta, and we have dropped all but the $J=1$ and $S=1$ terms, since only these are relevant here. m_s and m_s' denote the initial and final spin projections of the electron-positron system.

For the pair production vertex in Eq. (15), we have

$$\langle q_+q_-^{(+)}|j_\mu(0)|0\rangle=(4E_{q+}E_{q-})^{-\frac{1}{2}}(\bar{u}_{q-}|F_1^*(l^2)\gamma_\mu+F_2^*(l^2)\sigma_{\mu\nu}l_\nu|v_{q+}), \quad (17)$$

with $l_\mu=(q_++q_-)_\mu$. The complex conjugates of the form factors appear here because the outgoing boundary condition on the pair state changes the sign of the discontinuity of the matrix element across the branch cut. This may be easily seen by rederiving Eqs. (9) and (10) with the outgoing boundary condition. The change in the sign of the jump corresponds to replacing F by F^* according to reality condition Eq. (8).

In order to perform the angular integral remaining after the delta function is satisfied in Eq. (15), we must further reduce Eq. (17), explicitly exhibiting its angular variation in the center-of-mass system. We carry out this reduction by introducing the Pauli two-component spinors,

$$u_{q-}^{m-}=\begin{pmatrix}1\\ \boldsymbol{\sigma}\cdot\mathbf{q}/(E_{q-}+m_e)\end{pmatrix}\chi_{\frac{1}{2}}^{m-}(E_{q-}+m_e)^{\frac{1}{2}}$$

and $v_{q+}^{m+}=i\gamma_y u_{q+}^{m+*}$, in the representation with γ_y imaginary, transforming to the center-of-mass system $\mathbf{q}=\mathbf{q}_-=-\mathbf{q}_+=q\hat{q}$, $E_q=E_{q-}=E_{q+}$, and doing the spin algebra. Denoting the photon polarization unit vector by $e_\mu=(e_0,\hat{e})$ and introducing the unit vectors $\hat{e}_m=(-)^m\hat{e}_{-m}$ with components

$$\hat{e}_1=(-1,-i,0)/\sqrt{2},$$
$$\hat{e}_0=(0,0,1),$$

we obtain for production of the $e\bar{e}$ state with spin magnetic quantum number m_s,

$$\langle q_+q_-,m_s^{(+)}|j_\mu(0)|0\rangle e_\mu=\sqrt{2}\hat{e}_{m_s}^*\cdot\left[(F_1^*-4m_eF_2^*)\hat{e}+\left(-4F_2^*-\frac{F_1^*-4m_eF_2^*}{E_q+m_e}\right)\frac{q^2}{E_q}\hat{q}\hat{e}\cdot\hat{q}\right]$$
$$=\sqrt{2}\hat{e}_{m_s}^*\cdot\left[\hat{e}(F_1^*-4m_eF_2^*)-\hat{q}\hat{e}\cdot\hat{q}\left(1-\frac{m_e}{E_q}\right)(F_1^*+4E_qF_2^*)\right].$$

Carrying out the angular integral in Eq. (15) and performing the magnetic quantum number sums by straightforward application of identities for Clebch-Gordon coefficients, as given in the present notation in Appendix A of Blatt and Weisskopf,[10] we obtain

$$\mathrm{Im}\alpha=\frac{\mathrm{Re}T_{00}}{1-\mathrm{Im}T_{00}}\mathrm{Re}\alpha+\left(\frac{2}{9}\right)^{\frac{1}{2}}\frac{\mathrm{Re}(T_{02}\beta^*)}{1-\mathrm{Im}T_{00}},$$

$$\mathrm{Im}\beta=\left(\frac{2}{9}\right)^{\frac{1}{2}}\frac{\mathrm{Re}(T_{20}\alpha^*)}{1-(2/9)\,\mathrm{Im}T_{22}}+\frac{2}{9}\frac{\mathrm{Re}T_{22}}{1-(2/9)\,\mathrm{Im}T_{22}}\mathrm{Re}\beta, \quad (18)$$

where

$$\alpha(E)=\left(\frac{2E+m}{3E}\right)F_1-\left(\frac{E+2m}{3m}\right)4mF_2,$$

$$\beta(E)=\left(\frac{m-E}{E}\right)F_1-\left(\frac{E-m}{m}\right)4mF_2, \quad (19)$$

$$T_{\alpha\beta}=\frac{1}{2i}(S-1)_{\alpha\beta},$$

and

$$S=\begin{pmatrix}\cos^2\epsilon e^{2i\delta_0}+\sin^2\epsilon e^{2i\delta_2} & \frac{1}{2}\sin2\epsilon(e^{2i\delta_0}-e^{2i\delta_2})\\ \frac{1}{2}\sin2\epsilon(e^{2i\delta_0}-e^{2i\delta_2}) & \sin^2\epsilon e^{2i\delta_0}+\cos^2\epsilon e^{2i\delta_2}\end{pmatrix}, \quad (20)$$

in terms of the eigenphase shifts δ_0 and δ_2 for the $^3S_1$ and $^3D_1$ states and mixing parameter ϵ. The argument of δ_0, δ_2, and ϵ is E, the energy in the center-of-mass system, which is given by $q^2=4E^2$ in terms of the argument of the form factors F_1 and F_2.

Equations (18), (19), and (20) are in principle a complete solution to our problem in the approximation of keeping only the contribution of the $e\bar{e}$-intermediate state to the absorptive amplitude. In practice, however, this is not a very useful solution since we do not know δ_0, δ_2, and ϵ for electron-positron scattering. In order to obtain approximate values of these phase shifts, we return to Eq. (16) and calculate the $e\bar{e}$ scattering amplitude in renormalized Born approximation; i.e., we write

$$(2E_{p_-})^{-\frac{1}{2}}\langle p_+|\bar{u}_{p_-}j_e(0)|q_+q_-^{(+)}\rangle = e^2(16E_{p_+}E_{p_-}E_{q_+}E_{q_-})^{-\frac{1}{2}}\times\left\{\frac{(\bar{u}_{p_-}\gamma_\nu u_{q_-})(\bar{v}_{q_+}\gamma_\nu v_{p_+})}{(p_--q_-)^2}-\frac{(\bar{u}_{p_-}\gamma_\nu v_{p_+})(\bar{v}_{q_+}\gamma_\nu u_{q_-})}{(p_++p_-)^2}\right\}. \quad (21)$$

This is equivalent to retaining only the pole in the dispersion relations for $e\bar{e}$ scattering. It is then only necessary to insert Eqs. (21) and (17) into Eq. (15) and to perform the spin sums by standard trace techniques.

[10] J. M. Blatt and V. F. Weisskopf, *Theoretical Nuclear Physics* (John Wiley and Sons, Inc., New York, 1952).

There results from this approximate calculation of the absorptive amplitude the following expression:

$$\mathrm{Im}F_1\gamma_\mu+\mathrm{Im}F_2\sigma_{\mu\nu}q_\nu=\frac{e^2}{4\pi}\left(\frac{q^2-4m_e^2}{q^2}\right)^{\frac{1}{2}}\times\left\{(\mathrm{Re}F_1\gamma_\mu+\mathrm{Re}F_2\sigma_{\mu\nu}q_\nu)\tfrac{1}{4}\left(\frac{2q^2-4m_e^2}{q^2-4m_e^2}\right)\times\int_{-1}^{1}\frac{d\mu}{1-\mu}-\left(\frac{13}{12}\mathrm{Re}F_1\gamma_\mu+\mathrm{Re}F_2\sigma_{\mu\nu}q_\nu\right)-\tfrac{2}{3}\left(\frac{m_e^2}{q^2}\right)\mathrm{Re}F_1\gamma_\mu-\frac{m_e^2}{q^2-4m_e^2}(\mathrm{Re}F_1+8m_e\,\mathrm{Re}F_2)\times\left(\gamma_\mu+\frac{1}{4m_e}\sigma_{\mu\nu}q_\nu\right)\right\}. \quad (22)$$

The single angular integral which remains to be performed in Eq. (22) diverges logarithmically for $\mu\to1$. This divergence arises as follows: The Möller scattering amplitude becomes infinitely large for forward scattering because an infinite number of partial waves contribute in the limit $\mu=\cos\theta\to1$. The phase shifts themselves are also infinitely large in Born approximation, there being no finite partial-wave expansion of a scattering amplitude with angular variation $[\sin^2(\theta/2)]^{-1}$. In the vertex under consideration here, there are, however, only two eigenphase shifts for the $^3S_1$ and $^3D_1$ states appearing as we have seen in Eq. (20). In order to remove this divergence, then, we must give an exact treatment of the forward scattering amplitude. This it is possible to do because the two-body problem reduces to a one-body problem for forward scattering angles and the exact amplitude for scattering in the forward direction is available from analysis of the relativistic Coulomb scattering problem. That this is in fact so is easily seen if we transform to a coordinate system with one of the particles, say the positron, initially at rest. In the limit of small-angle scattering the positron remains at rest in this reference frame, does not flip its spin, and serves only as a point source of the Coulomb interaction.

We must also give an exact treatment of the scattering amplitude near threshold since the Born approximation is not valid as $q^2\to4m_e^2$, and makes the scattering amplitude become infinitely large there. It is also possible to do this because the two-body problem again reduces to a one-body problem at threshold and the exact phase shifts are available for Coulomb scattering.

We first show that an exact analysis of the forward scattering angles replaces the divergent expression by the following:

$$\tfrac{1}{2}\int_{-1}^{1}\frac{d\mu}{1-\mu}\to C=0.577\cdots, \quad (23)$$

the Euler-Mascheroni constant. The Born approximation amplitude, Eq. (21), is applicable for scattering angles $\theta > \theta_{\min}$ with $\theta_{\min} \ll 1$, but >0 so that only a finite number of phase shifts contribute. For $\theta < \theta_{\min}$ we replace Eq. (21) by the real part of the exact amplitude expressed here for convenience in the coordinate system in which the positron is at rest[11]:

$$\gamma_B = -\pi \frac{v_-}{p_-^2} \sum_l (-)^l \{ l^2 \,\mathrm{Im} C_l - (l+1)^2 \,\mathrm{Im} C_{l+1} \} P_l(\mu),$$

where to order $e^2/4\pi$,

$$\mathrm{Im} C_{l>0} = -\frac{e^2}{4\pi} \frac{1}{v_-} \frac{(-)^l}{l} \left[\frac{1}{l} + 2C - 2 \sum_{n=1}^{l} \frac{1}{n} \right].$$

Thus

$$\gamma_B = \frac{e^2}{4\pi} \frac{\pi}{p_-^2} \left\{ 2C - 1 + \sum_{l=1}^{\infty} 2(2l+1) \times \left[C - \sum_{n=1}^{l} \frac{1}{n} \right] P_l(\mu) \right\}. \quad (24)$$

We wish to join the contribution of Eq. (24) to the absorptive amplitude to the Born approximation contribution, Eq. (22), for $\mu > \theta_{\min}$. This requires the integration of $\gamma_B(\mu)$ over the angular interval $\mu = 1$ to $\mu = \cos\theta_{\min} \approx 1 - \theta_{\min}^2/2$. Since in fact only a finite number of phase shifts contribute, we are justified in interchanging the order of integration and summation in Eq. (24), obtaining[12]

$$\int_{1-\frac{1}{2}\theta_{\min}^2}^{1} \gamma_B d\mu = \frac{e^2}{4\pi} \frac{1}{2p_-^2} \left[2C - 1 - \sum_{l=1} P_l(\tfrac{1}{2}\theta_{\min}^2) \left(\frac{1}{l+1} + \frac{1}{l} \right) \right]. \quad (25)$$

The sum in Eq. (25) converges rapidly for $l > \theta_{\min}^{-1}$ and is readily evaluated, using the generating function for Legendre polynomials:

$$\sum_{l=1}^{\infty} P_l(\tfrac{1}{2}\theta_{\min}^2) \left(\frac{1}{l+1} + \frac{1}{l} \right) = 2 \ln\left(\frac{2}{\theta_{\min}} \right) - 1.$$

We obtain finally

$$\int_{1-\frac{1}{2}\theta_{\min}^2}^{1} \gamma_B d\mu = \frac{e^2}{4\pi p_-^2} \left(C - \ln \frac{2}{\theta_{\min}} \right) = \frac{e^2}{4\pi p_-^2} \left(C - \tfrac{1}{2} \int_{-1}^{\cos\theta_{\min}} \frac{d\mu}{1-\mu} \right). \quad (26)$$

[11] N. F. Mott and H. S. W. Massey, *Theory of Atomic Collisions* (Oxford University Press, Oxford, 1949), second edition, p. 80.

[12] E. T. Whittaker and G. N. Watson, *Modern Analysis* second edition, (Cambridge University Press, Cambridge, 1950), fourth edition, p. 333.

Since the forward Coulomb scattering amplitude is spin independent and since γ_B is the only rapidly varying function in Eq. (15) in the forward angular cone defined by $\theta_{\min}$, it is clear that the identification in Eq. (23) follows,[13] and we can now write from Eq. (22),

$$\mathrm{Im} F_1(q^2) = \frac{e^2}{4\pi} \left(\frac{q^2 - 4m_e^2}{q^2} \right)^{\frac{1}{2}} \times \left[\left\{ C - \frac{13}{12} - \frac{2m_e^2}{3q^2} + \frac{m_e^2(2C-1)}{q^2 - 4m_e^2} \right\} \times \mathrm{Re} F_1(q^2) - 8m_e \frac{m_e^2}{q^2 - 4m_e^2} \mathrm{Re} F_2(q^2) \right\},$$

$$\mathrm{Im} F_2(q^2) = \frac{e^2}{4\pi} \frac{1}{[q^2(q^2 - 4m_e^2)]^{\frac{1}{2}}} [-\tfrac{1}{4} m_e \,\mathrm{Re} F_1(q^2) + (C-1)(q^2 - 2m_e^2) \,\mathrm{Re} F_2(q^2)]. \quad (27)$$

We turn next to the threshold problem and note that at threshold ($E \to m$, $\epsilon \to 0$) Eqs. (18–20) imply

$$\mathrm{Im}(F_1 - 4m_e F_2) \to \tan\delta_0 \,\mathrm{Re}(F_1 - 4m_e F_2)$$

as

$$v_{\mathrm{c.m.}} = \left(\frac{q^2 - 4m_e^2}{q^2} \right)^{\frac{1}{2}} \to 0,$$

and Eq. (27) gives

$$\mathrm{Im}(F_1 - 4m_e F_2) \to \left(\frac{e^2}{4\pi} \right) \frac{C}{2v_{\mathrm{c.m.}}} \mathrm{Re}(F_1 - 4m_e F_2).$$

From the analysis of Coulomb scattering, it is known that

$$\tan\delta_0 = \left(\frac{e^2}{4\pi} \right) \frac{C}{v_{\mathrm{lab}}} = \left(\frac{e^2}{4\pi} \right) \frac{C}{2v_{\mathrm{c.m.}}} \quad \text{for} \quad v_{\mathrm{c.m.}} > \frac{4\pi}{e^2},$$

and that $\tan\delta_0$ goes through an infinite number of resonances with logarithmically increasing rapidity as $v_{\mathrm{c.m.}} \to 0$. In order to give the correct threshold behavior to the absorptive amplitude, it is necessary only to factor out $(e^2/4\pi)[q^2/(q^2 - 4m_e^2)]^{\frac{1}{2}}$ in Eq. (27) and make the substitution

$$\frac{e^2}{4\pi} \frac{1}{v_{\mathrm{c.m.}}} \to \frac{2}{C} \tan\delta_0.$$

[13] The correspondence of kinematic factors between Eqs. (22) and (26) comes out this way: putting amplitude (26) into (15) gives

$$\int d^3q_+ \int d^3q_- \,\delta^4(p_+ + p_- - q_+ - q_-) \frac{1}{p_-^2} = \int_{1-\frac{1}{2}\theta_{\min}^2}^{1} 2\pi d\mu \left(\frac{E p_-}{p_-} \right),$$

where $E p_-/p_-$ is expressed in the system with the positron at rest and may be written in invariant form as

$$\frac{p_- \cdot p_+}{[(p_- \cdot p_+)^2 - m_e^2]^{\frac{1}{2}}} = \frac{q^2 - 2m_e^2}{[q^2(q^2 - 4m_e^2)]^{\frac{1}{2}}}$$

corresponding to the coefficient of the logarithm in Eq. (22).

This leads to the equations

$$\mathrm{Im}F_1(q^2)=\left(\frac{2}{C}\tan\delta_0\right)\times\left[\left\{\frac{q^2-4m_e^2}{q^2}\left(C-\frac{13}{12}-\frac{2m_e^2}{3q^2}\right)+\frac{m_e^2}{q^2}(2C-1)\right\}\mathrm{Re}F_1-\frac{8m_e^2}{q^2}\mathrm{Re}F_2\right],\quad (27')$$

$$\mathrm{Im}F_2(q^2)=\left(\frac{2}{C}\tan\delta_0\right)\times\left[-\frac{m_e^2}{4q^2}\mathrm{Re}F_1+(C-1)\frac{q^2-4m_e^2}{q^2}\mathrm{Re}F_2\right].$$

Within the framework of our approximations, Eqs. (27′) represent the absorptive amplitudes for the electromagnetic vertex. Inserting them into dispersion integrals Eq. (6), we obtain the following integral equations for the electromagnetic form factors:

$$F_1(q^2)=\frac{1}{\pi}\int_{4m_e^2}^{\infty}\frac{dq'^2}{q'^2-q^2-i\epsilon}\times\left(\frac{2}{C}\tan\delta_0\right)\left(\left[\left\{C-\frac{13}{12}-\frac{2m_e^2}{3q'^2}\right\}\times\frac{q'^2-4m_e^2}{q'^2}+\frac{(2C-1)m_e^2}{q'^2}\right]\mathrm{Re}F_1(q'^2)-\left[\frac{8m_e^2}{q'^2}\right]\mathrm{Re}F_2(q'^2)\right),\quad (28)$$

$$F_2(q^2)=\frac{1}{\pi}\int_{4m_e^2}^{\infty}\frac{dq'^2}{q'^2-q^2-i\epsilon}\times\left(\frac{2}{C}\tan\delta_0\right)\left(-\frac{m_e}{4q'^2}\mathrm{Re}F_1(q'^2)+(C-1)\frac{q'^2-2m_e^2}{q'^2}\mathrm{Re}F_2(q'^2)\right).\quad (29)$$

Our discussion of the form factors is based on Eqs. (28) and (29).

IV. DISCUSSION

Let us first fix our attention on the moment form factor. The close agreement between the perturbation predictions of QED and experiment establish, to order $e^2/4\pi$, the moment value

$$F_2(0)=-\frac{1}{4\pi}\left(\frac{e^2}{4\pi}\right)\frac{e}{2m_e}=-\tfrac{1}{2}\Delta\mu_2$$

and the form factor variation

$$F_2(-|\mathbf{q}|^2)\cong-\tfrac{1}{2}\Delta\mu_2\left(\frac{2m_e^2}{|\mathbf{q}|^2}\ln\frac{|\mathbf{q}|^2}{4m_e^2}\right)$$

for space-like momentum transfers $m_e^2\ll|\mathbf{q}|^2\lesssim$ (several hundred Mev/c)^2. This behavior of $F_2(q^2)$ is reproduced by Eq. (29) if the charge form factor F_1 is replaced by its perturbation value,

$$F_1(q^2)\rightarrow F_1(0)=e,$$

for $q^2<(100\ \mathrm{Mev}/c)^2$. To this approximation the moment form factor in the right-hand side of Eq. (29) may be neglected since it is smaller in magnitude than F_1 and dies out sufficiently rapidly for large q values. Equation (29) then simplifies to

$$F_2(q^2)=\frac{1}{\pi}\int_{4m_e^2}^{\infty}\frac{dq'^2}{q'^2-q^2-i\epsilon}\left(\frac{2}{C}\tan\delta_0\right)\left(-\frac{em_e}{4q'^2}\right),\quad (30)$$

which agrees with experiment. Note also that the rms radius of the anomalous electron moment is $1/m_e$, the electron Compton wavelength.

Turning next to the charge form factor, we ask whether there exists a solution of Eq. (28) of the form used above for F_1 in fitting the observed moment structure. First of all, it is easy to see from Eq. (28) that there is no valid expansion of F_1 in powers of e^2, in contrast with the above discussion for F_2. Explicitly, if one attempts to write

$$F_1(q^2)=p_1(q^2)e^3+p_2(q^2)e^5+\cdots$$

the coefficient $p_1(q^2)$ is infinite. This reflects the fact that perturbation theory does not yield a charge form factor which satisfies a dispersion relation of type (A) with no subtraction, since, to lowest order in e, $F_1(q^2)=e$ [see Eq. (4)], a nonzero constant for $q^2\rightarrow\infty$. This is in violation of the assumption that the dispersion integral (A) exists. The subtracted form (B) of the dispersion relation, of course, provides a convergent expansion equivalent to the usual perturbation theory. Secondly, upon comparing the coefficients of Re $F_1(q^2)$ in Eqs. (28) and (29), we see that Eq. (28) for the charge form factor weights the region of high-momentum transfers by an extra power of (q'^2/m^2). In fact, since $(2/c)\tan\delta_0\rightarrow e^2/4\pi$ over almost the entire range of momentum values, Eq. (28) tells us on dimensional grounds that (q'^2/m^2) values up to $\sim\exp(137)$ will contribute. The neglect of all but the $e\bar{e}$ intermediate state in the absorptive amplitude is probably not valid at these large momentum transfers. Ignoring this for the moment, we may solve Eq. (28) in the following way. We assume that $mF_2(q^2)\ll F_1(q^2)$. Then

$$F_1(q^2)=\frac{1}{\pi}\int_{4m_e^2}^{\infty}\frac{dq'^2}{q'^2-q^2-i\epsilon}g_1(q'^2)\,\mathrm{Re}F_1(q'^2),\quad (31)$$

where

$$g_1(q^2)=\frac{2}{C}\tan\delta_0\left[\left(C-\frac{13}{12}-\frac{2m_e^2}{3q^2}\right)\times\frac{q^2-4m_e^2}{q^2}+(2C-1)\frac{m_e^2}{q^2}\right]. \quad (32)$$

Note that as $q'^2\to\infty$,

$$g_1(q^2)\to\left(\frac{e^2}{4\pi}\right)\left(C-\frac{13}{12}\right)<0. \quad (33)$$

According to Omnès,[14] the general solution to Eq. (31) is

$$F_1(q^2)=\frac{P(q^2)}{(q^2-4m_e^2)^n}\exp[\rho(q^2)], \quad (34)$$

where

$$\rho(q^2)=\frac{q^2}{\pi}\int_{4m_e^2}^{\infty}\frac{dq'^2}{q'^2(q'^2-q^2-i\epsilon)}\tan^{-1}g_1(q'^2), \quad (35)$$

n is an integer, and $P(q^2)$ is a polynomial.

P and n must be chosen so as to give the correct boundary conditions $F_1(0)=e$ and $F_1(\infty)=0$. The most appealing choice would be $n=0$, and $P(q^2)=e$. From Eq. (35), however, it is easy to see that

$$\rho(q^2)\to-\frac{1}{\pi}\tan^{-1}\left[\frac{e^2}{4\pi}\left(C-\frac{13}{12}\right)\right]\ln\left(\frac{q^2}{4m_e^2}\right)$$

as $q^2\to\infty$, and therefore with this choice of P and n, it is impossible to satisfy the condition $F_1(\infty)=0$. We trace this difficulty to the sign in Eq. (33). This is most unfortunate since this solution is practically a constant $(=e)$ for the range of $q^2/4m_e^2$ contributing to the moment form factor, and therefore reproduces perturbation theory.[15] It is as a result necessary to set n equal to 1. If this choice is made, the resulting F_1 is

$$F_1(q^2)=\frac{e}{1-(q^2/4m_e^2)}\exp[\rho(q^2)]. \quad (36)$$

It is possible to show that this results in an anomalous moment differing even in sign from the perturbation and experimental results. We conclude that within the approximations made, it is impossible to satisfy both the requirements that the solutions reproduce perturbation theory and that the form factors vanish at infinity. This is very likely due to the inadequacies of the approximations, particularly in the high-momentum region.

One may turn to the subtracted dispersion relation (B). For this case, the choice $n=0$ is acceptable, since all that is required is that $F_1(q^2)/q^2\to 0$ as $q^2\to\infty$. Thus, perturbation theory is reproduced by Eq. (B).

It is interesting to note in this connection that the particular subset of diagrams included here has changed the asymptotic behavior of the form factor from that indicated by perturbation theory. It is therefore not inconceivable that inclusion of all diagrams might produce form factors vanishing at infinity in support of the no-subtraction philosophy. It was to be hoped that the subset of diagrams actually included would itself be sufficient to produce vanishing form factors; this would have occurred if the sign in Eq. (33) had been reversed.

It may be worth remarking that if we accept the solution Eq. (34) with $n=0$ for low-momentum transfers, one obtains a mean-square radius

$$\langle r^2\rangle=\left(\frac{e^2}{4\pi}\right)\frac{1}{\pi}\left(2C-\frac{33}{20}\right)\frac{1}{m_e^2}<0. \quad (37)$$

The second moment of the charge distribution is thus observed to have a negative mean-square radius. It is of interest to recall here the analogous behavior of the photon or boson propagator in field theory.[16]

Finally, Eqs. (28) and (29) superficially look as though they might be an eigenvalue equation for the fine-structure constant $e^2/4\pi$. In fact, they are not, but have solutions for any value of $e^2/4\pi$. Equations (28) and (29) clearly cannot determine the charge e, since they are homogeneous in F_1 and F_2. As to whether the inclusion of other intermediate states in calculating the imaginary parts could determine e uniquely, we have no information.

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[14] R. Omnès (to be published).

[15] For large momenta it increases as

$$\exp\frac{1}{\pi}\left(\frac{13}{12}-c\right)\frac{e^2}{4\pi}\ln\frac{q^2}{4m_e^2}$$

and deviates from perturbation theory for $q^2>m_e^2e^{137}$. For such large values of momentum, Landau and collaborators have argued that electromagnetic processes must be damped out if $e^2/4\pi=1/137$ is to result from a consistent QED which starts from a real nonvanishing value of the bare charge e_0. See L. D. Landau, in *Niels Bohr and the Development of Modern Physics* (McGraw-Hill Book Company, Inc., New York, 1954).

[16] Schweber, Bethe, and de Hoffmann, *Mesons and Fields* (Row, Peterson, and Company, Evanston, 1955), Vol. I, p. 388.